

On asymptotic solutions of RFT in zero transverse dimensions

S. Bondarenko ^{a)}*, L. Horwitz ^{a),b)}†, J. Levitan ^{a)}‡, A. Yahalom ^{a)}§

^{a)} *Ariel University Center*

^{b)} *Tel Aviv University*

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Abstract

An investigation of dynamical properties of solutions of toy model of interacting Pomerons with triple vertex in zero transverse dimension is performed. Stable points and corresponding solutions at the limit of large rapidity are studied in the framework of given model. A presence of closed cycles in solutions is discussed as well as an application of obtained results for the case of interacting QCD Pomerons.

*Email: sergeyb@ariel.ac.il

†E-mail: larry@post.tau.ac.il

‡E-mail: levitan@ariel.ac.il

§Email: asher@fluidex-cfd.com

1 Introduction

A calculating of amplitudes of QCD high-energy scattering processes is a complex task based on different methods and approaches of high-energy physics, see [1, 2, 3, 4, 5]. In general this task is very complicated, and, therefore, the simplified models which have dynamics similar to QCD ones could serve as a good polygon for the different ideas and methods check. This is the reason why RFT-0 (Reggeon Field Theory in zero transverse dimensions) model is attracted much interest during few last years, see [6, 7]. The approach, which was formulated and studied a long time ago, even before the QCD era (see [8]), has a dynamics which is very similar to the solutions obtained in framework of interacting QCD Pomerons, see [9, 10].

In the paper [10] was shown, that we could understand the real RFT dynamics solving equations of motion of RFT-0. Quantum amplitudes in RFT-0 model framework are also calculable, see [7], and this is the only source for understanding, if only partially, of quantum dynamics of RFT based on QCD Pomerons. Nevertheless, even at classical level, there are some open questions which are interesting from point of view of QCD high-energy description of the process. These questions are about the asymptotic solutions of equations of motion, their stability and properties, see more details in [10]. In contrast to what we have in RFT in two transverse dimensions, the classical solutions of RFT-0 could be investigated analytically and achieved results could be applicable also in high-energy QCD.

The paper is organized as follows. In the next section we consider a way to approach asymptotic solutions of the model whereas in the Section 3 we analyze steady states of equations of motion. In Section 4 we discuss the interconnections between the steady-states and asymptotic solutions, relating the stable points of the equations with corresponding asymptotic solutions. Section 5 is dedicated to additional analysis of a stability of some asymptotic solutions. In Section 6 we investigate a possibility of periodic limited cycles of the solutions and Section 7 is a discussion of results and conclusion of the paper.

2 Asymptotic behavior of equation of motion

The Hamiltonian of our problem and corresponding equation of motions have the following form:

$$H = \mu q p - \gamma p^2 q - \gamma p q^2 \tag{1}$$

and

$$\dot{q} = \mu q - \gamma q^2 - 2\gamma p q \tag{2}$$

$$\dot{p} = -\mu p + \gamma p^2 + 2\gamma p q, \tag{3}$$

see [7, 8], where μ is an intercept of bare Pomeron and λ is a vertex of triple Pomeron interactions. The "time" variable of the equations is a rapidity, i.e. $\dot{q} = \frac{dq}{dy}$ with $y = \ln(s/s_0)$ and with s as a squared total energy of the process.. The boundary conditions are symmetric

$$q(y=0) = A, \quad p(y=Y) = A \quad (4)$$

where Y is the end of the "time" interval, i.e. Y is a total rapidity of the scattering process. We also request, that an amplitude of the "scattering" process, which depends on q and p variables, will preserve the so called "target-projectile" symmetry

$$Ampl.(q(y, A), p(y, A)) = Ampl.(q(Y - y, A), p(Y - y, A)) \quad (5)$$

see in [10] the definition of the scattering amplitude in the RFT.

In order to qualitatively understand behavior of the equations at large rapidities we rescale the rapidity and vertex of the problem:

$$y \rightarrow \mu y, \quad \lambda = \gamma / \mu \quad (6)$$

obtaining new equation of motion

$$\dot{q} = q - \lambda q^2 - 2\lambda p q \quad (7)$$

$$\dot{p} = -p + \lambda p^2 + 2\lambda p q. \quad (8)$$

We see, that in the rescaled equation the asymptotic limit for the rescaled total rapidity Y

$$Y_{Rescaled} = \mu Y \rightarrow \infty$$

could be achieved also by the limit

$$\mu \rightarrow \infty, \quad \gamma \propto \alpha_s^2 \rightarrow \infty$$

where also

$$\lambda = \gamma / \mu \propto \alpha_s \rightarrow \infty.$$

It means, that asymptotic solution of equations of motion for the case $Y \rightarrow \infty$ are the same as solutions in the limit $\lambda \rightarrow \infty$. Moreover, because our function must be analytical function of α_s , the solutions will be valid also when $Y \rightarrow \infty$ at finite λ . We note also, that due the initial condition Eq. (4), the asymptotic behavior of the $q = q(y, A)$ is achieved in the limit $y \rightarrow Y \rightarrow \infty$ whereas asymptotic of $p = p(y, A)$ function is lies in the limit $y \rightarrow 0$ with $Y \rightarrow \infty$.

The rescaled Hamiltonian of the problem now has the following form:

$$H = qp - \lambda p^2 q - \lambda p q^2 = E \quad (9)$$

with equation of motion

$$\dot{q} = q - \lambda q^2 - 2\lambda p q \quad (10)$$

$$\dot{p} = -p + \lambda p^2 + 2\lambda p q \quad (11)$$

where

$$y \rightarrow \mu y \quad E \rightarrow E/\mu$$

in equations of motion and Hamiltonian ¹. The solution of the equation Eq. (9) for variables q and p at the limit of large λ are two pairs of the functions:

$$q_{1,2} = \left(\frac{1}{\lambda} - p - \frac{E}{\lambda p^2}, \frac{E}{\lambda p^2} \right) \quad (12)$$

and

$$p_{1,2} = \left(\frac{1}{\lambda} - q - \frac{E}{\lambda q^2}, \frac{E}{\lambda q^2} \right). \quad (13)$$

Further, as any pair of asymptotic solution $(q, p)_i$ of equation motion Eq. (10) - Eq. (11), we will consider the pairs (q_i, p_i) $i = 1..2$ which contain functions Eq. (12) - Eq. (13).

3 Steady states of equations of motion

The definition of steady states of the equation of motion is a standard:

$$q (1 - \lambda q - 2\lambda p) = 0 \quad (14)$$

$$-p (1 - \lambda p + 2\lambda q) = 0 \quad (15)$$

Due the fact of absence of rapidity dependence in these equations we assume that at least part of solutions (steady states) of these equations describe the solutions of equation of motion in the asymptotic limit $Y \rightarrow \infty$, whereas $y \rightarrow Y$ for the $q = q(y, A)$ and $y \rightarrow 0$ for the $p = p(y, A)$

The non-trivial steady states of the equations could be easily found

$$(q_{St}, p_{St})_1 = \left(\frac{1}{\lambda}, 0 \right) \quad (16)$$

$$(q_{St}, p_{St})_2 = \left(0, \frac{1}{\lambda} \right) \quad (17)$$

$$(q_{St}, p_{St})_3 = \left(\frac{1}{3\lambda}, \frac{1}{3\lambda} \right) \quad (18)$$

In general, the linearized analysis of the equations of motion around these states could be performed.

The Jacobian matrix

$$J = \begin{pmatrix} 1 - 2\lambda q - 2\lambda p & -2\lambda q \\ 2\lambda p & -1 + 2\lambda q + 2\lambda p \end{pmatrix} \quad (19)$$

¹Further, we shell use everywhere the sign E instead E/μ .

lead to the following eigenvalues for these three steady states

$$\lambda_1^{12} = \pm 1 \quad \lambda_2^{12} = \pm 1 \quad \lambda_3^{12} = \pm \frac{i}{\sqrt{3}}. \quad (20)$$

Unfortunately, the fact that, for example, the first two steady states are saddles does not say a lot about particular corresponding asymptotic solutions which also must satisfy boundary conditions Eq. (4). Therefore, in the next two section we will study these steady states in correspondence with the solutions Eq. (12) - Eq. (13) in the sense of convergence of asymptotic solutions to the corresponded steady state in the limit of asymptotically large rapidity. In the case of the convergence existing we could call a steady state as a stable point.

4 The $(q_{St}, p_{St})_1$ and $(q_{St}, p_{St})_2$ steady states and corresponding solutions of equation of motion

In this section we compare the solutions Eq. (12) - Eq. (13) with the results of the steady states analysis Eq. (16) - Eq. (17). We note, that the energy, which corresponds to these steady states is zero:

$$E = (pq - \lambda p q^2)_{\{(q_{St}, p_{St})_1, (q_{St}, p_{St})_2\}} = 0. \quad (21)$$

Therefore, in the limit $E \rightarrow 0$ a correspondence between these states and the asymptotic solution is established easily:

$$(q_{St}, p_{St})_1 = \left(\frac{1}{\lambda}, 0 \right) = (q_1, p_2)_{E=0} \quad (22)$$

and correspondingly

$$(q_{St}, p_{St})_2 = \left(0, \frac{1}{\lambda} \right) = (q_2, p_1)_{E=0}. \quad (23)$$

In order to find the asymptotic solution which corresponds to the steady states/stable points of Eq. (22) (we will look for the solution for the states of Eq. (22), the solution for Eq. (23) could be found similarly) we will consider the equation Eq. (7) as the equation consisting of two parts. The first part provides a fully integrable equation, whereas the second part, $2\lambda q p$, is a small perturbation around $p \propto 0$. Keeping only leading terms in both Eq. (7) - Eq. (8) we arrive, therefore, to the following system of equations:

$$\dot{q} = q - \lambda q^2 \quad (24)$$

$$\dot{p} = -p + 2\lambda p q. \quad (25)$$

The consistency of the approximation we check thereafter by taking asymptotic limits in the pair (q, p) of solution and comparison of obtained result with conditions Eq. (22) or Eq. (23).

Integration of the system Eq. (24)- Eq. (25) with given initial conditions Eq. (4) leads to the following functions:

$$q(y, A) = \frac{A e^y}{A \lambda (e^y - 1) + 1} \quad (26)$$

$$p(y, A) = A e^{Y-y} \left(\frac{A \lambda (e^y - 1) + 1}{A \lambda (e^Y - 1) + 1} \right)^2. \quad (27)$$

In order to find the asymptotic behavior of the solutions we take the limit $Y \rightarrow \infty$ and finite λ . We see, that in this limit, where $y \rightarrow Y \rightarrow \infty$ for q and $y \rightarrow 0, Y \rightarrow \infty$ for p , we obtain

$$q(y, A) = \frac{A e^y}{A \lambda (e^y - 1) + 1} \xrightarrow{y \rightarrow Y \rightarrow \infty} \frac{1}{\lambda} \quad (28)$$

$$p(y, A) = A e^{Y-y} \left(\frac{A \lambda (e^y - 1) + 1}{A \lambda (e^Y - 1) + 1} \right)^2 \xrightarrow{y \rightarrow 0, Y \rightarrow \infty} \frac{e^{-Y}}{A \lambda^2} \rightarrow 0. \quad (29)$$

These asymptotic values coincide with the asymptotic values (q_1, p_2) from Eq. (12)- Eq. (13) and stable points Eq. (16). As expected, taking firstly the limit $\lambda \rightarrow \infty$, we will come to the same stable point Eq. (12)- Eq. (13):

$$q(y, A) = \frac{A e^y}{A \lambda (e^y - 1) + 1} \xrightarrow{\lambda \rightarrow \infty, y \rightarrow Y \rightarrow \infty} \frac{1}{\lambda} \quad (30)$$

$$p(y, A) = A e^{Y-y} \left(\frac{A \lambda (e^y - 1) + 1}{A \lambda (e^Y - 1) + 1} \right)^2 \xrightarrow{\lambda \rightarrow \infty, y \rightarrow 0, Y \rightarrow \infty} A e^{-Y} \rightarrow 0 \quad (31)$$

in the limit $Y_{Rescaled} \propto \alpha_s Y \rightarrow \infty$.

The second solution, which corresponds to the stable point $(q_{St}, p_{St})_2$ could be found similarly. It has the following form

$$q(y, A) = A e^y \left(\frac{A \lambda (e^{Y-y} - 1) + 1}{A \lambda (e^Y - 1) + 1} \right)^2 \quad (32)$$

$$p(y, A) = \frac{A e^{Y-y}}{A \lambda (e^{Y-y} - 1) + 1}. \quad (33)$$

We also see, that an amplitude consisting of the pairs of solutions, Eq. (26)- Eq. (27) and Eq. (32)- Eq. (33), will preserve the symmetry of the problem Eq. (5)

5 The $(q_{St}, p_{St})_3$ steady state and corresponding solution of equation of motion

We search an asymptotic solution which characterized by following steady state:

$$(q_{St}, p_{St})_3 = \left(\frac{1}{3\lambda}, \frac{1}{3\lambda} \right). \quad (34)$$

As a pair of asymptotic solutions (q, p) , which could corresponds to this state, we take the following combination

$$(q_1, p_1) = \left(\frac{1}{\lambda} - p - \frac{E}{\lambda p^2}, \frac{1}{\lambda} - q - \frac{E}{\lambda q^2} \right), \quad (35)$$

and our further task is a check of the self-consistency of this assumption. The Hamiltonian for this steady state/stable point is not zero:

$$H(q = 1/3\lambda, p = 1/3\lambda) = E = \frac{1}{27\lambda^2}, \quad (36)$$

so we need to find solutions of equations of motion which provide this non-zero energy.

Let's consider full equation for the q variable and, as we did previously above, we will expand the last "perturbative" term in the equation taking there $p = p_1$ from Eq. (35)

$$\dot{q} = q - \lambda q^2 - 2\lambda p q = q - \lambda q^2 - 2\lambda p_1 q \quad (37)$$

We obtain

$$\dot{q} = q - \lambda q^2 - 2\lambda p_1 q = q - \lambda q^2 - 2\lambda q \left(\frac{1}{\lambda} - q - \frac{E}{\lambda q^2} \right) = -q + \lambda q^2 + \frac{2E}{q}. \quad (38)$$

Again, considering the last term as an perturbation, we take there $q = q_{St}$ from Eq. (34) and rescaling rapidity $\mu y \rightarrow \mu \lambda y$ we get finally:

$$\dot{q} = q^2 - q/\lambda + 6E. \quad (39)$$

The solution of this equation with the initial condition given by Eq. (4) is

$$q = \frac{1}{2\lambda} + \dot{E} \frac{A - 1/(2\lambda) + \dot{E} + e^{2\dot{E}y} (A - 1/(2\lambda) - \dot{E})}{A - 1/(2\lambda) + \dot{E} - e^{2\dot{E}y} (A - 1/(2\lambda) - \dot{E})} \quad (40)$$

where

$$\dot{E} = \sqrt{\frac{1}{4\lambda^2} - 6E} \quad (41)$$

and rapidity is γy . In the asymptotic limit $y \rightarrow Y \rightarrow \infty$ we obtain a simple expression

$$q = \frac{1}{2\lambda} - \dot{E}. \quad (42)$$

For the p variable we similarly obtain

$$p = \frac{1}{2\lambda} + \dot{E} \frac{A - 1/(2\lambda) + \dot{E} + e^{2\dot{E}(Y-y)} (A - 1/(2\lambda) - \dot{E})}{A - 1/(2\lambda) + \dot{E} - e^{2\dot{E}(Y-y)} (A - 1/(2\lambda) - \dot{E})} \quad (43)$$

that in the asymptotic limit of variable p , which is $y \rightarrow 0, Y \rightarrow \infty$, gives

$$p = \frac{1}{2\lambda} - \dot{E}. \quad (44)$$

How, for the verification of the self-consistency of the solution, we back to the asymptotic pair of solutions Eq. (35). We have

$$q = \frac{1}{2\lambda} - \dot{E} = \frac{1}{\lambda} - p - \frac{E}{\lambda p^2}. \quad (45)$$

Taking value of p from Eq. (44) and value of E from Eq. (41) we obtain

$$-2\lambda \dot{E} \left(\frac{1}{2\lambda} - \dot{E} \right)^2 = \dot{E}^2/6 - 1/(24\lambda^2). \quad (46)$$

Among three solutions of this equation there is a solution

$$\dot{E} = \frac{1}{6\lambda} \quad (47)$$

which gives

$$(q_1, p_1) = \left(\frac{1}{2\lambda} - \dot{E}, \frac{1}{2\lambda} - \dot{E} \right)_{\dot{E}=\frac{1}{6\lambda}} = \left(\frac{1}{3\lambda}, \frac{1}{3\lambda} \right) = (q_{St}, p_{St})_3 \quad (48)$$

and

$$E = 1/(24\lambda^2) - \dot{E}^2/6 = \frac{1}{27\lambda^2} = H(q = 1/3\lambda, p = 1/3\lambda). \quad (49)$$

Clearly, the required amplitude's symmetry Eq. (5) for the Eq. (40), Eq. (43) functions is also preserved.

Thereby we proved that our solution is self-consistent ant that solution Eq. (35) (i.e. functions Eq. (40), Eq. (43)) is related with the Eq. (18) stable point. Other pairs of asymptotic solutions for $E \neq 0$, such as (q_2, p_2) , (q_1, p_2) and (q_2, p_1) , do not satisfy the condition of correspondence of their large rapidity limits with the steady state Eq. (18).

6 Stability analysis and Lyapunov function

Obtained asymptotic solutions are stable in the sense of their behavior at asymptotically large rapidity. Nevertheless, it is interesting to check the asymptotic stability of the solution from the point view of Lyapunov function, namely, we will try to construct the local Lyapunov functions for each stable point and corresponding asymptotic solution obtaining the stability criteria from this side of the problem.

We will look for the local Lyapunov functions, it means that we will search for the functions which have following properties:

1. first property:

$$F(q, p) \geq 0 \quad (50)$$

for the q and p defined in neighborhood region of stable point;

2. second property:

$$F(q, p)_{q=q_{SP}, p=p_{SP}} = 0; \quad (51)$$

3. third property:

$$\left(\frac{dF(q, p)}{dy} \right) < 0 \quad (52)$$

in neighborhood region of stable point.

6.1 Lyapunov function for Eq. (16)- Eq. (17) stable points

We consider stable points Eq. (16)

$$(q_{St}, p_{St})_1 = \left(\frac{1}{\lambda}, 0 \right) \quad (53)$$

and corresponding asymptotic solution (q_1, p_2) defined by Eq. (26)- Eq. (27). For the stable point $(q_{St}, p_{St})_1$ we consider a following function as a candidate for the Lyapunov function:

$$F(q, p) = H - p^n + \lambda p^2 q \quad (54)$$

where H is the Hamiltonian Eq. (9) and $n > 3$. Now we will check properties of this function as a Lyapunov one.

1. We look for the behavior of the Lyapunov function around the stable point and we have at leading order at asymptotically large Y :

$$F(q, p)_{(q_1, p_2)} = (qp - \lambda p q^2 - p^n)_{(q_1, p_2)} = \frac{e^{-2Y} (1 - \lambda A)}{A^2 \lambda^4} > 0, \quad (55)$$

the condition $n > 3$ is used here for the derivation of the answer and where we assume $\lambda A < 1$.

2. Checking second request for the Lyapunov function we obtain

$$F(q = \frac{1}{\lambda}, p = 0) = 0 \quad (56)$$

simply by definition of the function which is proportional to p .

3. Third property of our function is the following

$$\left(\frac{dF(q, p)}{dy} \right)_{(q_1, p_2)} = (qp - \lambda p q^2 - p^n)'_{(q_1, p_2)} = (\dot{H}_0 - n p^{n-1} \dot{p})_{(q_1, p_2)}. \quad (57)$$

Here $H_0 = qp - \lambda p q^2$ is a "fan" Hamiltonian with precise solution of the corresponding equations of motion in the form functions Eq. (26)- Eq. (27). Therefore, we have $\dot{H}_0 = 0$ and obtain:

$$\left(\frac{dF(q, p)}{dy} \right)_{(q_1, p_2)} = -(n p^{n-1} \dot{p})_{(q_1, p_2)} = -n p^n (-1 + \lambda p + 2 \lambda q)_{(q_1, p_2)}. \quad (58)$$

In leading order, around $(q_1 = \frac{1}{\lambda}, p_2 = \epsilon)$, we obtain:

$$\left(\frac{dF(q, p)}{dy} \right)_{(q_1, p_2)} = -n \epsilon^n = -n \frac{e^{-nY}}{A^n \lambda^{2n}} < 0. \quad (59)$$

where the value of ϵ is taken from Eq. (29). Thereby we see, that function Eq. (54) is the Lyapunov function for the stable point Eq. (53). A Lyapunov function for the stable point Eq. (17) could be constructed similarly.

6.2 Lyapunov function for Eq. (18) stable point

For the stable points Eq. (18)

$$(q_{St}, p_{St})_3 = \left(\frac{1}{3\lambda}, \frac{1}{3\lambda} \right) \quad (60)$$

and corresponding asymptotic solution Eq. (40) and Eq. (43) as a Lyapunov we consider the following function

$$F(q, p) = \ln \left(H / (\lambda q^2 p) \right) \quad (61)$$

where H is the Hamiltonian of Eq. (9). This function has following properties.

1. In the neighborhood of the stable point, where $q = \frac{1}{3\lambda} - \epsilon$ and $p = \frac{1}{3\lambda} - \epsilon$ with $\epsilon < 1$, see Eq. (40) and Eq. (43), we have in leading order on ϵ :

$$F(q, p) = \ln(1 + 9\epsilon\lambda) \approx 9\epsilon\lambda > 0 \quad (62)$$

with

$$\epsilon = \frac{e^{-Y/3} A - 1/3\lambda}{3\lambda A - 2/3\lambda}. \quad (63)$$

It is interesting to note also, that for

$$1/3 < A\lambda < 2/3 \quad (64)$$

condition Eq. (62) for the function Eq. (61) is not satisfied.

2. Taking precise value of the stable point Eq. (18) we see that

$$F(q = \frac{1}{3\lambda}, p = \frac{1}{3\lambda}) = \ln 1 = 0. \quad (65)$$

3. Taking the derivative of this function over rapidity we obtain:

$$\left(\frac{dF(q, p)}{dy} \right) = \dot{H} - 2 \frac{\dot{q}}{q} - \frac{\dot{p}}{p}. \quad (66)$$

At leading order on ϵ we have that $\dot{H} = 0$ and we obtain

$$\left(\frac{dF(q, p)}{dy} \right) = -2(1 - \lambda q - 2\lambda p) - (-1 + \lambda p + 2\lambda q). \quad (67)$$

At leading order we, therefore, have

$$\left(\frac{dF(q, p)}{dy} \right) = (-1 + 3\lambda q)_{q=\frac{1}{3\lambda}-\epsilon} = -3\epsilon < 0. \quad (68)$$

Thereby we see, that our function Eq. (61) is a Lyapunov function for the stable point Eq. (18).

7 Periodic limited cycles of solution

In order to show the presence of periodic cycles in asymptotic solutions of our equations, we back to the original Hamiltonian Eq. (1)

$$H = \mu q p - \gamma p^2 q - \gamma p q^2 \quad (69)$$

and determines the vertices of equation correspondingly to the QCD Pomeron vertices :

$$\mu = \alpha_s, \quad \gamma = \alpha_s^2, \quad (70)$$

where α_s is QCD coupling constant, see [6]. Introducing new rescaled fields

$$p \rightarrow \frac{p}{\alpha_s}, \quad q \rightarrow \frac{q}{\alpha_s}, \quad (71)$$

we obtain rescaled Hamiltonian:

$$H = \frac{1}{\alpha_s} (q p - p^2 q - p q^2). \quad (72)$$

Now we follow the results of article [11]. Consider the system of three variables q, p and z :

$$\begin{aligned} \dot{q} &= q (1 - q - \alpha p - \beta z) \\ \dot{p} &= p (1 - p - \beta q - \alpha z) \\ \dot{z} &= z (1 - z - \alpha q - \beta y) \end{aligned} \quad (73)$$

with additional condition $\alpha + \beta = 2$. The dynamics of following function

$$F = p + q + z \quad (74)$$

could be described with the use of equations Eq. (73):

$$\frac{dF}{dy} = F (1 - F). \quad (75)$$

At $Y \rightarrow \infty$ the solution of this equation is simply

$$F \rightarrow 1, \quad p + q + z \rightarrow 1. \quad (76)$$

Therefore, at asymptotically large rapidity Y possible solutions of Eq. (73) lie on the triangle Eq. (76), i.e. all orbits at large enough rapidity set up on this triangle. Further we project this motion on the $q - p$ plane by use of projection

$$z = 1 - q - p \quad (77)$$

which has the place at large rapidity in system Eq. (73). Rewriting the equations Eq. (73) we obtain for $q - p$ plane dynamics:

$$\dot{q} = \frac{\alpha - \beta}{2} q (1 - q - 2p) \quad (78)$$

$$\dot{p} = -\frac{\alpha - \beta}{2} p (1 - p - 2q) . \quad (79)$$

We see, that these equations could be considered as equations of motion for the following Hamiltonian:

$$H = \frac{\alpha - \beta}{2} (qp - p^2q - pq^2) . \quad (80)$$

Comparing Eq. (80) and Eq. (69) we conclude, therefore, that our Hamiltonian of interests could be considered as projection of dynamics of the system of three variables Eq. (73) on the plane $q - p$ in the limit of large rapidity Y with the

$$\alpha = 1 + \frac{1}{\alpha_s} \quad (81)$$

$$\beta = 1 - \frac{1}{\alpha_s} . \quad (82)$$

We conclude, therefore, that at large rapidity limit the dynamics of the system of two variables Eq. (7) and system of three variables Eq. (73) is the same. The systems are dual in the limit of large Y .

In order to illustrate the periodic solutions in Eq. (73), and therefore periodic solutions in Eq. (7), we introduce an additional function:

$$W = qpz . \quad (83)$$

The dynamics of this function is described by following equation:

$$\frac{d}{dy} \ln(W) = 3 - 3W = 3 \frac{d}{dy} \ln(F) , \quad (84)$$

that gives after the integration:

$$W(y) = W_0 \left(\frac{F(y)}{F(0)} \right)^3 \xrightarrow{y \rightarrow \infty} W_0 \left(\frac{1}{F(0)} \right)^3 = C_0 . \quad (85)$$

We see, that overall dynamics at large Y is restricted by the following conditions

$$p + q + z = 1 \quad (86)$$

and

$$pqz = C_0 . \quad (87)$$

Conditions imposed on the solutions of the Eq. (73) at large rapidity result as cycle orbits, which are intersection of the triangle Eq. (86) and hyperboloid Eq. (87). In this case, the projection Eq. (77), determines also cycle dynamics on the plane defined by equations of motion Eq. (7).

The duality of the systems Eq. (7) and Eq. (73) could be also understood from the point of conditions needed to apply on the systems in order to obtain the solution. In the case of system Eq. (7) we need the boundary conditions Eq. (4) and value of the energy of the system E . In the

²It is interesting to note that the Eq. (73) system is drastically simplified in the limit $\alpha_s \rightarrow \infty$

case of equation Eq. (73) we need only boundary (initial) conditions $(q_0, p_0, z_0,)$. Excluding z from Eq. (77) and Eq. (87) we obtain:

$$p q (1 - q - p) = C_0 = E. \quad (88)$$

We see that the following relation for the dual systems exists:

$$E = C_0 = \frac{q_0 p_0 z_0}{q_0 + p_0 + z_0}. \quad (89)$$

Hereby we conclude, that the energy parameter in system Eq. (7) is determined by the initial value z_0 in the system Eq. (73) at given (q_0, p_0) .

In our particular example for stable points Eq. (22) at $E = 0$ we see, that $p_0(y = 0, Y \rightarrow \infty) \rightarrow 0$ provides $E = C_0 = 0$ for corresponding asymptotic solution Eq. (26)- Eq. (27) through Eq. (89). In this case, in order to satisfy Eq. (86)- Eq. (87) we also need to conclude that $z(Y \rightarrow \infty) \rightarrow 0$. In general, therefore, whole dynamics of the system Eq. (7) at large rapidity and at $E = 0$ is described by the line

$$q(y) + p(y) = 1 \quad (90)$$

with stable points given by Eq. (16)- Eq. (17). In the case when $E \neq 0$ at the limit of large Y , the cycles are not lines anymore but they are cycles determined by Eq. (86)- Eq. (87).

8 Conclusion

In this paper we investigated the form and behavior of solutions of RFT-0 theory at the limit of asymptotically large rapidities. We developed a method which allows to find an analytical form of asymptotic solution basing on the analysis of steady states of the equations of motion of corresponding Hamiltonian. Found solutions satisfy the boundary conditions of the problem, Eq. (4), and preserve the symmetry of the problem, Eq. (5), and therefore, they could be considered also as the approximate solutions of our equations of motion in the limit of large rapidity. The results obtained are important because solutions of RFT based on QCD Pomerons exhibits similar properties and have similar structure due the similarities of equations of motion, see [10].

The pairs of the functions Eq. (26)-Eq. (27) and Eq. (32)-Eq. (33) were known a long time ago, see [12], as "fan" diagrams solutions. They are good approximation for the scattering amplitude in the case when the scattering situation is not symmetrical, see [9, 10]. Therefore, it was interesting to check, whether the same functions provide minimal energy for the Hamiltonian also in the case of symmetrical scattering with conditions Eq. (4), see "fan" dominance effect discussion in [10]. Our results, thereby, prove that in the semi-classical approximation the "fan" diagrams are the leading contribution into the scattering amplitude at the limit of large rapidity also in the case of scattering

of symmetrical objects. The stability of these solutions was shown as well by use of the usual stability analysis, namely by the Lyapunov functions construction, see Eq. (54).

Considering the symmetrical solution Eq. (40)-Eq. (43) we see that this solution is suppressed in the amplitude, see [10], in comparison with the non-symmetrical, providing non-zero Hamiltonian. Therefore, in calculations of different observables of RFT based on QCD Pomerons, [9, 10], in the first approximation the leading contribution comes from the "fan" diagrams. The symmetrical solution could be considered, therefore, as sub-correction to the amplitude on the semi-classical level. The stability of this solution is checked by the construction of Lyapunov function Eq. (61), which was obtained in assumption that condition Eq. (64) is not satisfied. Is this a sign of instability of this solution for some values of parameters is not clear.

Another result of our investigation is a presence of limit cycles solutions which could be demonstrated on the base of the system of equations dual to the considered one in the limit of large rapidity. In order to understand possible consequences of such solutions on observables of scattering processes in RFT at high energies an additional investigation is required. Also, in the future studies, it will be important to understand a mechanism of reduction of the limit cycle to the one from the stable points.

A problem of an origin of three solutions during the evolution of the system with rapidity is a subject which was not fully considered in the paper. A simplest analysis shows, that new solution are arose when the the function p or q begins to be large enough. It happens approximately at rapidity $y \propto -\frac{1}{\mu} \ln(\lambda A) \rightarrow 1$, but we have no dynamical description of the process similar to usual bifurcation picture. It have be noted also, that more complicated boundary conditions lead to more complicated picture of possible solutions of the equations of motion, see [13]. A rise of these solutions it is an interesting problem which we plan investigate in our further work as well, as the problems mentioned above.

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